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General parameter relations for the Shinnar-Le Roux pulse design algorithm

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Abstract

The magnetization ripple amplitudes from a pulse designed by the Shinnar-Le Roux algorithm are a non-linear function of the Shinnar-Le Roux A and B polynomial ripples. In this paper, the method of Pauly et al. [J. Pauly, P. Le Roux, D. Nishimura, A. Macovski, Parameter relations for the Shinnar-Le Roux selective excitation pulse design algorithm, IEEE Transactions on Medical Imaging 10 (1991) 56–65.] has been extended to derive more general parameter relations. These relations can be used for cases outside the five classes considered by Pauly et al., in particular excitation pulses for flip angles that are not small or 90°. Use of the new relations, together with an iterative procedure to obtain polynomials with the specified ripples from the Parks–McClellan algorithm, are shown to give simulated slice profiles that have the desired ripple amplitudes.

Keywords: RF pulse design; Shinnar-Le Roux

1. Introduction

The design of selective excitation pulses has been greatly simplified by the Shinnar-Le Roux (SLR) algorithm [1–5]. In the algorithm, the RF pulse is first approximated as a series of hard pulses, each giving rise to a small rotation of the magnetization vector. Via the spinor notation, the total rotation caused by the pulse can be described by two z-transform polynomials, usually referred to as the SLR A and B polynomials [5]. This is known as the forward SLR transform. The reverse procedure is the inverse SLR transform, which generates the required RF pulse from given A and B polynomials, designed according to the desired slice profile.

Pauly et al. described a method of obtaining the required polynomials using filter design and the Parks–McClellan (PM) algorithm [5]. First, the slice profile is characterized by stop and pass bands. From the stopband, passband, and amplitude ripples in each band, the transition width is estimated, and these parameters are passed to the PM algorithm. The algorithm returns a polynomial with an equi-ripple response which, after scaling, is taken as the B polynomial. Next, the A polynomial is derived from the B polynomial. Given the A and B polynomials, the RF pulse is obtained through the inverse SLR transform. Unfortunately, the magnetization ripple amplitudes in the slice profile, δ^e, are a non-linear function of the polynomial ripple amplitudes, δ. The relationship between δ^e and δ was derived in Pauly et al. for five types of pulses: small-tip-angle and 90° excitation pulses, inversion pulses, crushed spin–echo, and 90° suppression pulses. These are reproduced in Table 1 and will be referred to as the Pauly parameter relations.

While these expressions are very useful, they do not cover the entire range of flip angles, and may not be optimal for certain cases. For example, neither the small-tip-angle or 90° relations for excitation pulses are exactly right for a flip angle of say, 70°. In their paper, Raddi and Klose [6] observed that this lack of a general parameter relation for different flip angles led Matson, in his public-domain software MATPULSE [7], to use the Pauly relations outside their design range; for example, using the 90° relation for excitation pulses with $\phi = 45$–$135°$. Raddi and Klose

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Table 1
Exact and Pauly parameter relations for various RF pulses

Pulse type	Parameter of interest	Pauly relations		Exact relations	
		δ_1^e	δ_2^e	δ_1^e	δ_2^e
Small-tip-angles	M_{xy}	δ_1	δ_2	Eq. (11)	Eq. (12)
90°	M_{xy}	$2\delta_1^2$	$\sqrt{2}\delta_2$	$\sqrt{2-(1+\delta_1)^2}(1+\delta_1)-1$	$\sqrt{2-\delta_2^2}\delta_2$
Inversion	M_z	$8\delta_1$	$2\delta_2^2$	$8\delta_1/(1+\delta_1)^2$	$2\delta_2^2/(1+\delta_1)^2$
Crushed spin–echo	M_{xy}	$4\delta_1$	δ_2^2	$4\delta_1/(1+\delta_1)^2$	$\delta_2^2/(1+\delta_1)^2$
Suppression	M_z	$2\delta_1$	δ_2^2	$2\delta_1+\delta_1^2$	δ_2^2

derived approximate parameter relations for transverse and longitudinal magnetization, before combining them to produce a generalized parameter relation between polynomial and magnetization ripple. This generates a weighted value of δ dependent on flip angle only, regardless of pulse type. The limitation of this approach is that the parameter of interest is often either the transverse ripple *or* the longitudinal ripple. For the inversion and suppression pulses, the longitudinal ripple is estimated, whereas for the excitation and spin–echo pulses it is the transverse ripple that is estimated. Combining both may in certain cases be suboptimal. The most satisfactory approach is to use more general parameter relations pertaining to the transverse or longitudinal ripple, but covering a wider range of angles.

Pauly et al. did not use the same approach in deriving the relations for the different pulse types. For excitation pulses, a geometrical argument was used, while a more general approach was used for the other pulse types. The general approach is straightforward, and when applied to the excitation pulses, gives the required general parameter relations.

2. Method

The forward SLR transform on a pulse comprising n hard pulses separated in time by $\triangle t$, applied during a slice select magnetic field gradient G, results in the two z-transform polynomials of order $n-1$, the A and B polynomials [5]. Their magnitudes are related by:

$$|A_n|^2+|B_n|^2=1 \tag{1}$$

The A and B polynomials can be used to calculate a Bloch simulation of the pulse on magnetization (M_x,M_y,M_z) (if relaxation can be ignored), through the following matrix equation [6]:

$$\begin{pmatrix} M_{xy}^+ \\ M_{xy}^{+*} \\ M_z^+ \end{pmatrix} = \begin{pmatrix} z^{-n}(A_n^*)^2 & -z^{-n}B_n^2 & 2A_n^*B \\ -z^{-n}(B_n^*)^2 & z^nA_n^2 & 2A_nB_n^* \\ -z^{-n}A_n^*B_n^* & -z^nA_nB_n & 1-2B_nB_n^* \end{pmatrix} \begin{pmatrix} M_{xy}^- \\ M_{xy}^{-*} \\ M_z^- \end{pmatrix} \tag{2}$$

where $z=e^{i\gamma Gx\triangle t}$, $M_{xy}=M_x+iM_y$, and the $-$ and $+$ superscripts denote 'before' and 'after' the pulse.

If the initial magnetization is in the z-direction only, as is usual when considering excitation pulses, then:

$$M_{xy}^+=2A_n^*B_nM_z^- \tag{3}$$

$$M_z^+=(1-2B_nB_n^*)M_z^- \tag{4}$$

If the initial magnetization is in the y-direction, as is usual when considering spin–echo pulses, then the magnitude slice profile after a spin–echo pulse surrounded by crusher gradients is given by [5]:

$$|M_{xy}^+|=|B_n^2M_y^-| \tag{5}$$

Given the specifications for the ripple inside the passband δ_1, ripple in the stopband δ_2, pulse duration and slice width, the optimum transition width W is obtained via an empirically derived formula (Eq. (20) in Ref. [5]). The parameters δ_1, δ_2 and W are passed as inputs into the PM algorithm, which then returns a polynomial with an equi-ripple response in the pass and stop bands, normalized to have an average value of 1 in the passband (Fig. 6 in Ref. [5]). When correctly scaled, this polynomial is taken as the B polynomial. This also determines the magnitude of the A polynomial through Eq. (1).

The rules for scaling are as follows: for flip angles ϕ such that $|(1+\delta_1)\sin(\phi/2)|\leqslant 1$, the polynomial is scaled by multiplying with the factor $i\sin(\phi/2)$. This gives, for instance, the usual $M_{xy}^+=M_z^-\sin\phi$ and $M_z^+=M_z^-\cos\phi$ as the average in-slice magnetization. After scaling, the magnitude of the passband maximum becomes $(1+\delta_1)\sin(\phi/2)$ and that of the stopband becomes $\delta_2\sin(\phi/2)$. This factor cannot be used where $|(1+\delta_1)\sin(\phi/2)|>1$ because from Eq. (4) or (5), after scaling, the maximum value of the B polynomial results in $|M|>1$, which is physically incorrect. Instead, the polynomial is scaled by multiplying with $i/(1+\delta_1)$ to make the maximum $|B_n|=1$. For example, in the case of $\phi=180°$, where the condition is always true, the magnitude of the minimum in the passband becomes $(1-\delta_1)/(1+\delta_1)$, and the magnitude of the maximum in the stopband becomes $\delta_2/(1+\delta_1)$.

These maximum (or minimum) and average values can now be simply substituted into Eqs. (3)–(5) and subtracted from each other to give the ripples in the magnetization component of interest. Essentially, this was how Pauly et al. derived the parameter relations for inversion, crushed spin–echo and 90° suppression pulses. As an example, the derivation is repeated here for the in- and out-slice ripples, δ_1^e and δ_2^e, of a suppression pulse. It differs from the original in that this result applies not only to $\phi=90°$, but more generally, where ϕ is such that $|(1+\delta_1)\sin(\phi/2)| \leqslant 1$. The parameter of interest is M_z; using Eq. (4):

$$\begin{aligned}\delta_1^e &= M^+_{z,\max} - M^+_{z,\text{ave}} \\ &= \left[1 - 2(1+\delta_1)^2\sin^2(\phi/2)\right] - \left[1 - 2\sin^2(\phi/2)\right] \\ &= 2\sin^2(\phi/2)\left(-\delta_1^2 - 2\delta_1\right) \end{aligned} \tag{6}$$

Note that M_z^- has been set to 1 for convenience. Similarly for out-slice ripple:

$$\begin{aligned}\delta_2^e &= M^+_{z,\max} - M^+_{z,\text{ave}} = \left[1 - 2\delta_2^2\sin^2(\phi/2)\right] - 1 \\ &= -2\delta_2^2\sin^2(\phi/2)\end{aligned} \tag{7}$$

In the case considered by Pauly et al., of $\phi = 90°$,

$$\delta_1^e = -\delta_1^2 - 2\delta_1 \tag{8}$$

$$|\delta_1^e| \approx 2\delta_1 \tag{9}$$

and

$$|\delta_2^e| = \delta_2^2 \tag{10}$$

which are the Pauly parameter relations of Table 1.

This approach was not used by Pauly et al. for excitation pulses, where a geometrical approach was used instead. Furthermore, only expressions for small flip angles and 90° were derived. However, by using the general approach, exact expressions applicable to a wider range of angles may be obtained. The parameter of interest is now taken to be the ripple amplitude in the transverse magnetization; the derivation above is repeated, but with M_{xy}. Only the case of flip angle for which $|(1+\delta_1)\sin(\phi/2)| \leqslant 1$ is relevant practically as the other case corresponds to little or no transverse magnetization:

$$|\delta^e| = \left|\left(M^+_{xy,\max} - M^+_{xy,\text{ave}}\right)/M^+_{xy,\text{ave}}\right|$$

$$|\delta_1^e| = \left|\frac{2\sqrt{1-(1+\delta_1)^2\sin^2(\phi/2)}(1+\delta_1)\sin(\phi/2) \mp \sin\phi}{\sin\phi}\right| \tag{11}$$

$$|\delta_2^e| = \left|\frac{2\sqrt{1-\delta_2^2\sin^2(\phi/2)}\delta_2\sin(\phi/2)}{\sin\phi}\right| \tag{12}$$

The sign change in Eq. (11) takes into account the change from $\phi < 180°$ to $>180°$, and the expressions have been normalized to the average in-slice magnetization.

In the case of $\phi = 90°$, these equations reduce to:

$$|\delta_1^e| = \left|\sqrt{2-(1+\delta_1)^2}(1+\delta_1) - 1\right| \tag{13}$$

$$|\delta_2^e| = \left|\sqrt{2-\delta_2^2}\delta_2\right| \tag{14}$$

By repeating the derivation for the other cases considered by Pauly et al., the corresponding exact expressions were obtained and are reproduced in Table 1. Note that, following Pauly et al., the definition of ripple in inversion and crushed spin–echo cases differs slightly from the other cases in that: $\delta^e = |M^+_{\max} - M^+_{\min}|$ instead of $|M^+_{\max} - M^+_{\text{ave}}|$.

Eqs. (11) and (12) can be graphed (see Figs. 1 and 2) and inverted numerically. In Fig. 3, the exact in-slice parameter relation for an excitation pulse is graphed for $\phi = 70°$, and compared with the Pauly relations for small-tip-angle and 90°. Note that the exact parameter relation is not symmetric: as the flip angle approaches 90°, the positive ripple contributes less to the magnetization ripple. In this case of $\phi \leqslant 90°$, the maximum in-slice ripple δ_1^e is obtained by considering the negative δ_1.

Given the parameter relations in Table 1, it is still necessary to generate polynomials with the required $\delta_{1,2}$. It was found that using the ripple and transition width suggested in Ref. [5] for inputs into the PM algorithm often gave ripple amplitudes that did not match the specified ripples. Therefore, the input ripples into the PM algorithm were refined iteratively, until a match was obtained. For example, suppose the initial δ_2 input results in a polynomial

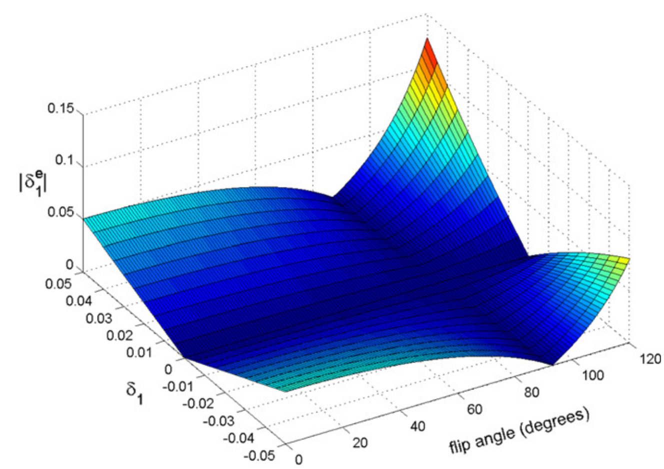

Fig. 1. $|\delta_1^e|$ (normalized to average in-slice magnetization) as a function of flip angle and δ_1.

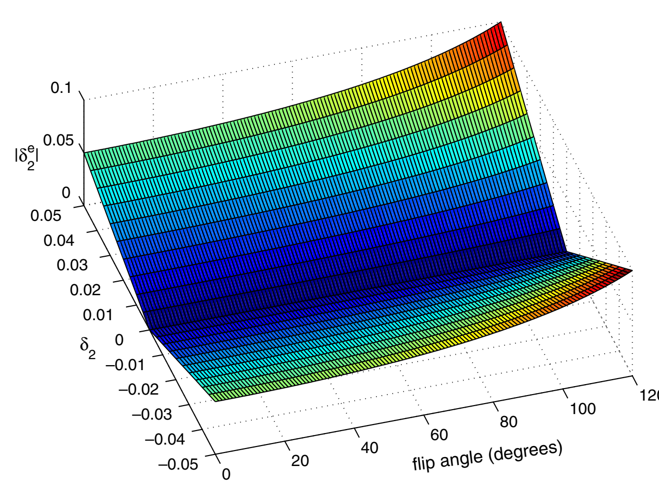

Fig. 2. $|\delta_2^e|$ (normalized to average in-slice magnetization) as a function of flip angle and δ_2.

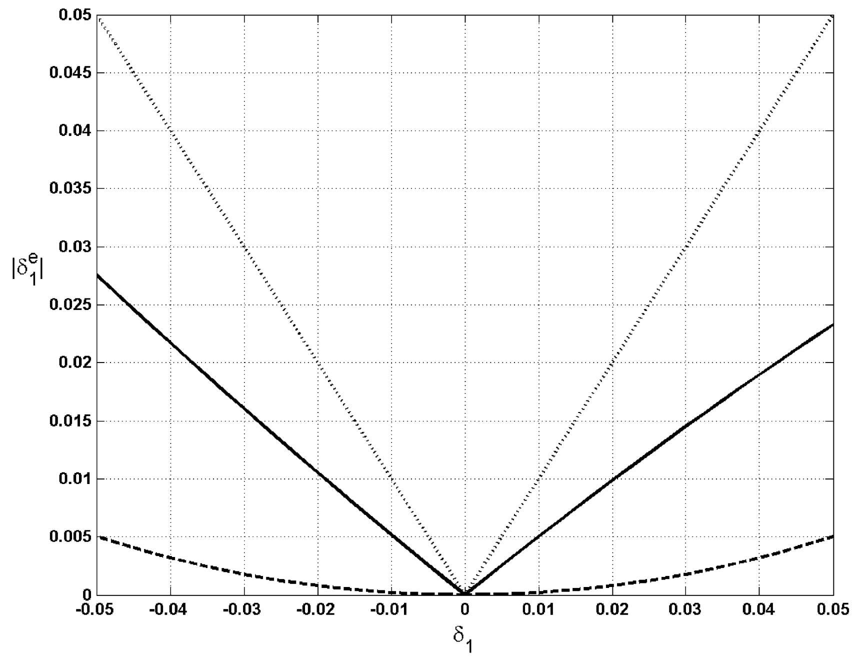

Fig. 3. Comparison of parameter relations for $|\delta_1^e|$ at $\phi = 70°$: general (solid), Pauly small-tip-angle (dotted), Pauly 90° (dashed).

ripple that is too small. The input δ_2 is incremented by a small value (a thousandth of the initial value was used). Next, in an inner iterative loop, the MATLAB routine *remezord*, which uses the same estimate as used in Ref. [5], is used to find the minimum transition width consistent with the incremented δ_2 (plus unchanged input δ_1) and the order of the polynomial. In effect, the width of the transition band is altered to obtain the required ripple amplitudes. These values are passed to the PM algorithm and the δ_2 from the resulting polynomial measured. In this way, the input δ_2 is slowly increased (with the minimum W found at each stage) until the measured δ_2 is within 1% of the desired value. Finally, the input δ_1 is adjusted in the same way. Following adjustment of δ_1, δ_2 will not necessarily still be correct, but it was usually found to be near enough. For the results below, δ_1 and δ_2 were each refined only once.

The algorithms were implemented using locally developed code in MATLAB. Within MATLAB, the PM algorithm was implemented using the *remez* function from the Signal Processing Toolbox Version 6.0(R13). Excitation pulses were designed using δ's obtained using the general parameter relations and iterative refinement of ripple amplitudes, and their slice profiles were compared to those obtained using the method and relations described by Pauly et al. [5].

3. Results

Results for two excitation pulse examples are presented. For a flip angle of 70°, slice thickness = 2 kHz, ripple amplitudes = 1% for both $\delta_{1,2}$, and pulse duration 4 ms, defined over 100 points, generalized relations gave: $\delta_1 = 0.0190$, $\delta_2 = 0.0081$. Initial use of the PM algorithm with these and the optimum transition width $W = 21\%$ (via Eq. (21) in Ref. [5]) resulted in a polynomial with $\delta_1 = 0.0247$, $\delta_2 = 0.0106$, which were unsatisfactory. Input δ's were iteratively refined to $\delta_1 = 0.0151$, $\delta_2 = 0.0064$, and $W = 23\%$, which gave the required ripples. Table 2 shows these input parameters together with those from Pauly small-tip-angle and 90° excitation. Table 2 also shows the magnetization ripples obtained from Bloch simulation of the resulting pulses, using the public-domain software MATPULSE [7]. Figs. 4 and 5 show the simulated in-slice and out-slice profiles of the three designs—using generalized relations gives almost exactly the required ripples.

For a flip angle of 110°, but with other parameters unchanged from above, generalized relations gave: $\delta_1 = 0.0091$, $\delta_2 = 0.0057$. With these values and the optimum $W = 25\%$, the PM algorithm gave a frequency response with $\delta_1 = 0.0120$, $\delta_2 = 0.0076$, which were also unsatisfactory. Refinement of ripples gave $\delta_1 = 0.0091$, $\delta_2 = 0.0058$, obtained with inputs of $\delta_1 = 0.0071$, $\delta_2 = 0.0044$, and $W = 27\%$. The simulated magnetization was compared with that obtained using the 90° relation. Results are summarized in Table 3. Figs. 6 and 7 show the simulated in-slice and out-slice profiles. The values in

Table 2
Parameters for 70° excitation pulse, showing inputs into PM algorithm, and resulting (normalized) magnetization ripples

	Input			Output	
	W (%)	δ_1 (%)	δ_2 (%)	δ_1^e (%)	δ_2^e (%)
General	23	1.51	0.64	0.97	0.99
Small-tip-angle	23	1.00	1.00	0.63	1.51
90°	17	7.07	0.71	3.90	0.94

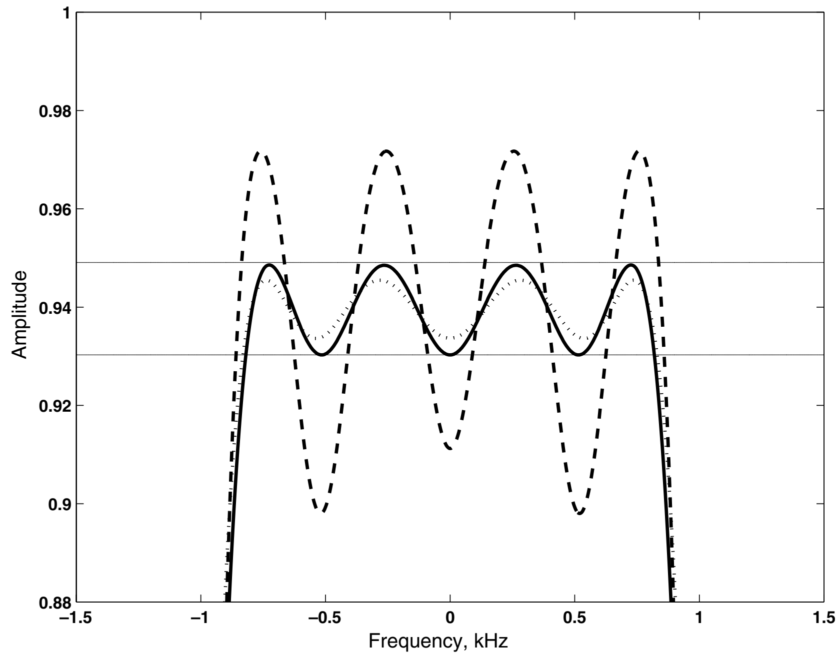

Fig. 4. In-slice magnetization ripple at $\phi = 70°$: general relation (solid), Pauly small-tip-angle (dotted), Pauly 90° (dashed).

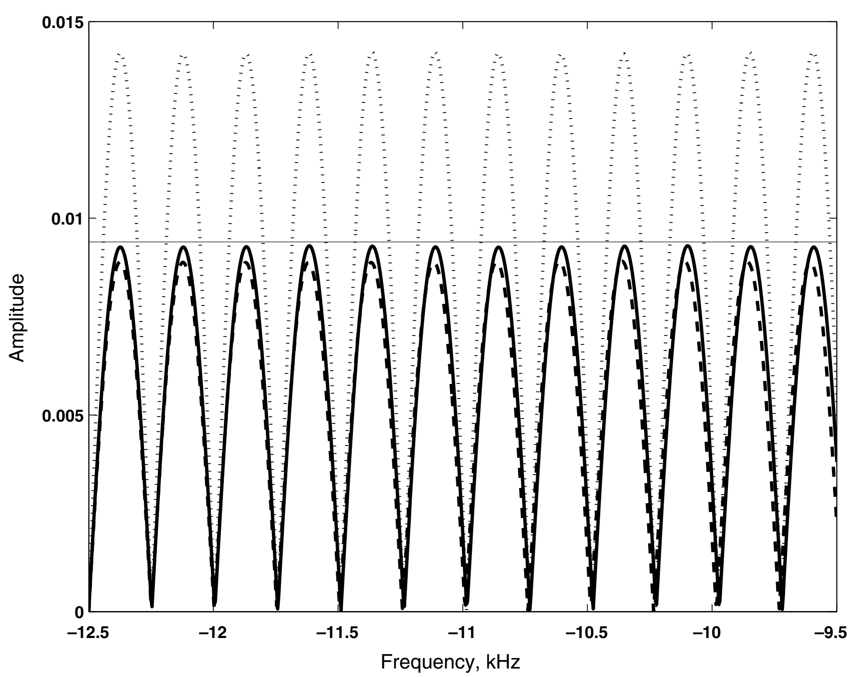

Fig. 5. Out-slice magnetization ripple at $\phi = 70°$: general relation (solid), Pauly small-tip-angle (dotted), Pauly 90° (dashed).

Table 3
Parameters for 110° excitation pulse, showing inputs into PM algorithm, and resulting (normalized) magnetization ripples

	Input			Output	
	W (%)	δ_1 (%)	δ_2 (%)	δ_1^e (%)	δ_2^e (%)
General	27	0.71	0.44	0.98	1.00
90°	17	7.07	0.71	5–12	1.35

the transition band cannot be controlled, but outside it, the generalized relations give the required ripple.

4. Discussion and conclusion

In the pulse design method of Pauly et al., the resulting magnetization ripple is determined by two stages: first, the

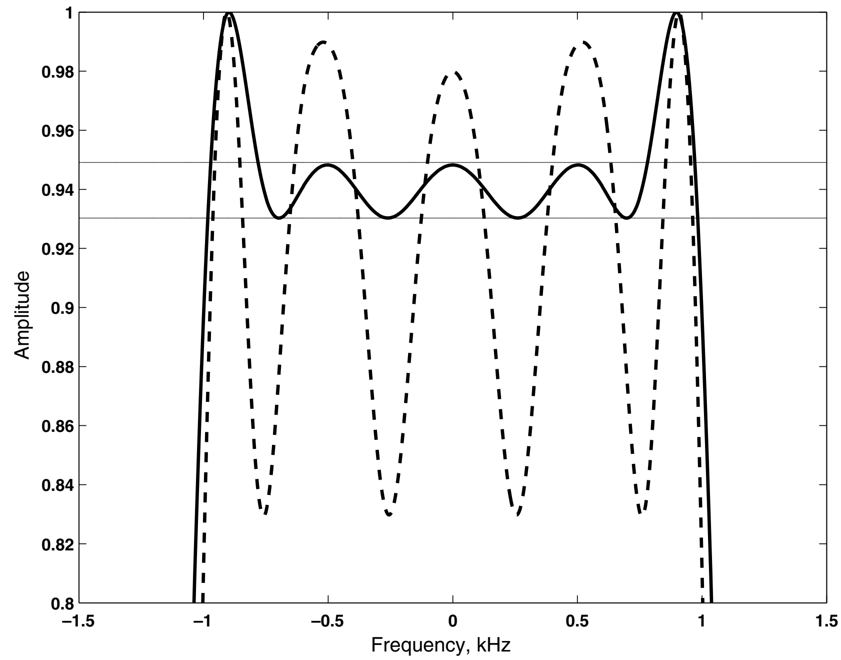

Fig. 6. In-slice magnetization ripple at $\phi = 110°$: general relation (solid), Pauly 90° (dashed).

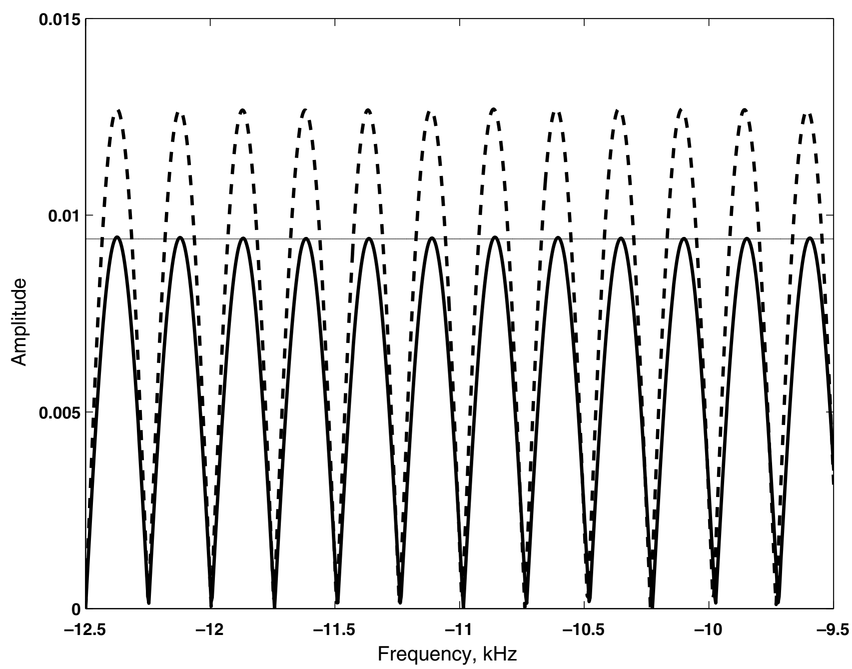

Fig. 7. Out-slice magnetization ripple at $\phi = 110°$: general relation (solid), Pauly 90° (dashed).

relations between the polynomial and magnetization ripple, and second, the generation of a polynomial with the required ripple. In this paper, the method of Pauly et al. has been extended to obtain exact parameter relations for a wider range of angles. Although the extension is straightforward, it does not appear to be widely known. The expressions derived in this way in Table 1 can be shown to be very nearly equal to the Pauly relations, and therefore it might not seem advantageous to have the more exact formulas. However, they are useful for the class of excitation pulses that were not considered in Ref. [5] i.e., those with flip angles not limited to small angles or 90°. They might also be useful, for example, where transmit pulse amplitude variation results in a range of flip angles. The resulting general relations are more cumbersome than the Pauly relations, but they are simple to invert numerically.

The polynomials generated by the PM algorithm were often found not to have the specified ripples. Thus, a heuristic method of adjusting the input parameters of the PM algorithm was used to obtain polynomials with the desired δ's. This, in combination with the new relations, produced magnetization ripples that match almost exactly the required ripples, and show an improvement in performance compared with the Pauly relations for flip angles not previously considered in Ref. [5]. However, as there is an inverse relationship between ripple magnitude and transition width, achieving the correct ripple may be at the cost of increasing the transition width.

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